TUTORIAL - 2

1. The pdf of a "r" is $f(x) = \begin{cases} kx(2-x) & , 0<x<2 \\ 0 & , \text{otherwise} \end{cases}$.

Find k:

W.k.t

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} kx(2-x)\,dx = 1$$

$$k\int_{0}^{2} (2x - x^2)\,dx = 1$$

$$k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[2^2 - \frac{2^3}{3}\right] = 1$$

$$k\left(4 - \frac{8}{3}\right) = 1$$

$$k\left(\frac{12-8}{3}\right) = 1$$

$$k\left(\frac{4}{3}\right) = 1$$

$$k = \frac{3}{4}.$$

$$\therefore f(x) = \begin{cases} \frac{3}{4}x(2-x) & , 0<x<2 \\ 0 & , \text{otherwise} \end{cases}$$

Find the r^{th} moment about the origin

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^2 x^r \frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_0^2 x^{r+1}(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right) dx$$

$$= \frac{3}{4}\left(\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right)_0^2$$

$$= \frac{3}{4}\left(\frac{2(2^{r+2})}{r+2} - \frac{2^{r+3}}{r+3} - 0 + 0\right)$$

$$= \frac{3}{4}\left(\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right)$$

$$= \frac{3}{4} \times 2^{r+3} \left(\frac{1}{r+2} - \frac{1}{r+3} \right)$$

$$= \frac{3}{4} \times 2^{r+1} \times 2^2 \left(\frac{r+3-r-2}{(r+2)(r+3)} \right)$$

$$= 3 (2^{r+1}) \left(\frac{1}{(r+2)(r+3)} \right)$$

$$\mu_r' = E(x^r) = 3 \cdot \frac{2^{r+1}}{(r+2)(r+3)}$$

Find the first 4 moments

$$r=1, \ \mu_1' = 3 \cdot \frac{2^{1+1}}{(1+2)(1+3)} = \cancel{3} \cdot \frac{2^2}{3 \times 4} = 1$$

$$r=2, \ \mu_2' = 3 \cdot \frac{2^{2+1}}{(2+2)(2+3)} = 3 \cdot \frac{2^3}{4 \times 5} = \frac{3 \times 8^2}{4 \times 5} = \frac{6}{5}$$

$$r=3, \ \mu_3' = 3 \cdot \frac{2^{3+1}}{(3+2)(3+3)} = \cancel{3} \cdot \frac{2^{4}}{5 \times \cancel{6}} = \frac{8}{5}$$

$$r=4, \ \mu_4' = 3 \cdot \frac{2^{4+1}}{(4+2)(4+3)} = \cancel{3} \cdot \frac{2^5}{\cancel{6} \times 7} = \frac{16}{7}$$

2. Let x be a r.v with PDF $f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

Find MGF

w.k.t. $M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

$$= \int_0^{\infty} e^{tx} \frac{1}{3} e^{-x/3} dx = \frac{1}{3} \int_0^{\infty} e^{-\left(\frac{1}{3}-t\right)x} dx$$

$$= \frac{1}{3} \left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)} \right]_0^{\infty} = \frac{1}{3} \left[\frac{e^{-\infty}}{-\left(\frac{1}{3}-t\right)} - \frac{e^0}{-\left(\frac{1}{3}-t\right)} \right]$$

$$= \frac{1}{\cancel{3}} \left[0 + \left(\frac{1}{\frac{1-3t}{\cancel{3}}} \right) \right] = \frac{1}{3} \left(0 + \frac{1}{\left(\frac{1}{3}-t\right)} \right) = \frac{1}{\cancel{3}} \left(\frac{1}{\frac{1-3t}{\cancel{3}}} \right)$$

$$M_x(t) = \frac{1}{1-3t}$$

$$P(x>3) = \int_3^\infty f(x)\,dx = \int_3^\infty \frac{1}{3} e^{-\frac{x}{3}}\,dx$$

$$= \frac{1}{3}\int_3^\infty e^{-x/3}\,dx = \frac{1}{3}\left[\int_3^\infty e^{-(\frac{1}{3})x}\,dx\right]$$

$$= \frac{1}{3}\left[\frac{e^{-(1/3)x}}{-1/3}\right]_3^\infty = \frac{1}{3}\left[\frac{e^{-\infty}}{-1/3} - \frac{e^{-(\frac{1}{3})3}}{-1/3}\right]$$

$$= \frac{1}{3}\left[0 + \frac{e^{-1}}{+1/3}\right] = \frac{1}{3}(3e^{-1})$$

$$= e^{-1}$$

Mean : $E(x) = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{1}{1-3t}\right)\right]_{t=0}$

$$= \left[\frac{-1}{(1-3t)^2} \times (-3)\right]_{t=0}$$

$$= \left[\frac{3}{(1-3t)^2}\right]_{t=0}$$

$$E(x) = 3$$

Variance $E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{d}{dt} \frac{3}{(1-3t)^2}\right]_{t=0}$

$$= 3\left[\frac{-2}{(1-3t)^3} \times (-3)\right]_{t=0}$$

$$= 3\left[\frac{6}{(1-3t)^3}\right]_{t=0}$$

$$E(x^2) = 18$$

3. If $f(x) = \begin{cases} x e^{-x^2/2} & , x>0 \\ 0 & , \text{otherwise} \end{cases}$, show that $f(x)$ is valid or not.

We have to prove $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} x e^{-x^2/2} dx$$

Let $\frac{x^2}{2} = t$

differentiate

$\frac{\cancel{2}x dx}{\cancel{2}} = dt$

$x\,dx = dt$

x	0	∞
t	0	∞

$$= \int_0^\infty e^{-t}\,dt$$

$$= (e^{-t})_0^\infty$$

$$= (e^{-\infty} + e^{0})\Big]_0$$

$$= 0 + 1$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Hence $f(x)$ is a valid pdf.

4. If the density function of a continuous RV is

$$f(x) = \begin{cases} ax & , 0 < x < 1 \\ a & , 1 < x < 2 \\ 3a - ax & , 2 < x < 3 \\ 0 & , \text{otherwise} \end{cases}$$

1) find a.

w.k.t. $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + \cancel{\int} = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + a\left[3x - \frac{x^2}{2}\right]_2^3 = 1$$

$$a\left(\frac{1}{2} - 0\right) + a(2-1) + a\left(\left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)\right) = 1$$

$$a\left[\frac{1}{2} + 1 + \left(\frac{9}{2} - \frac{8}{2}\right)\right] = 1$$

$$a\left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1$$

$$a(2) = 1$$

$$a = \frac{1}{2}$$

ii) Find CDF

$$f(x) = \begin{cases} x/2 & , 0 < x < 1 \\ 1/2 & , 1 < x < 2 \\ 3/2 - x/2 & , 2 < x < 3 \\ 0 & , \text{otherwise} \end{cases}$$

ii) Find CDF

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

In the interval $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x$$

$$F(x) = \frac{x^2}{4}$$

In the interval $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \frac{1}{2}\int_{0}^{1} x\,dx + \frac{1}{2}\int_{1}^{x} dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0\right) + \frac{1}{2}(x-1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$F(x) = \frac{x}{2} - \frac{1}{4}$$

In the interval $2 \le x \le 3$

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\int_{0}^{1} x\,dx + \frac{1}{2}\int_{1}^{2} dx + \frac{1}{2}\int_{2}^{x} (3-x)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^x$$

$$= \frac{1}{2}\left(\frac{1}{2}-0\right) + \frac{1}{2}(3-2) + \frac{1}{2}\left(3x - \frac{x^2}{2} - 6 + 2\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - \frac{4}{2}$$

$$\frac{1}{4} + \frac{1}{2} - 2$$

$$\frac{1+2-8}{4} = \frac{-5}{4}$$

$$F(x) = \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

In the interval $x > 3$

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx + \int_{3}^{x} 0\,dx$$

$$= \frac{1}{2}\int_{0}^{1} x\,dx + \frac{1}{2}\int_{1}^{2} dx + \frac{1}{2}\int_{2}^{3} (3-x)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^3$$

$$= \frac{1}{2}\left(\frac{1}{2}-0\right) + \frac{1}{2}(2-1) + \frac{1}{2}\left(\left(9-\frac{9}{2}\right) - (6-2)\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\left(\frac{9}{2} - 4\right)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$$

$$F(x) = 1$$

$$\therefore F(x) = \begin{cases} x^2/4 & , 0 \le x \le 1 \\ x/2 - 1/4 & , 1 \le x \le 2 \\ 3x/2 - x^2/4 - 5/4 & , 2 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

5. Find the r^{th} moment about the origin for the RV with pdf $f(x) = \begin{cases} kx^2 e^{-x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$. Hence find

Sol: To find "k"

By the definition, we've $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} kx^2 e^{-x}\,dx = 1$$

$$k\int_{0}^{\infty} x^2 e^{-x}\,dx = 1$$

$$k\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k\left[0 - (-2)\right] = 1$$

$$2k = 1$$

$u = x^2 \quad (+) \quad v = e^{-x}$

$u' = 2x \quad (-) \quad v_1 = \frac{e^{-x}}{-1}$

$u'' = 2 \quad (+) \quad v_2 = \frac{e^{-x}}{(-1)^2}$

$u''' = 0 \quad v_3 = \frac{e^{-x}}{-1}$

$$K = \frac{1}{2}$$

$$\therefore f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

r^{th} moment about the origin

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \left(\frac{1}{2}x^2 e^{-x}\right) dx$$

$$= \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^{r+2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^{(r+3)-1}\,dx$$

$$\mu_r' = \frac{1}{2}\Gamma(r+3)$$